QM-HW4

司书屹 2022011090

- **3.1** 算符$\hat{A}$是厄米算符的要求是

  $$(\psi,\hat{A}\phi)=(\hat{A}\psi,\phi)$$

  将F_+带入左式，由厄米共轭性质$(\hat{A}\psi,\phi)=(\psi,\hat{A}^\dagger\phi)$，有

  $$\begin{aligned}(\psi,F_+\phi)&=\frac{1}{2}[(\psi,F\phi)+(\psi,F^\dagger\phi)]\\&=\frac{1}{2}[(F^\dagger\psi,\phi)+(F\psi,\phi)]\\&=(F_+\psi,\phi)\end{aligned}$$

  同理可证F_+,F_-均为厄米算符。另外，通过简单带入可以证明

  $$F=\frac{1}{2}(F+F^+)+i\frac{1}{2i}(F-F^+)=F_++iF_-$$

- **3.2**
  - $\boldsymbol{l}=\boldsymbol{r}\times\boldsymbol{p}$

    $$\boldsymbol{l}=\begin{vmatrix}i&j&k\\x&y&z\\p_x&p_y&p_z\end{vmatrix}\Rightarrow\begin{cases}l_x=yp_z-zp_y\\l_y=zp_x-xp_z\\l_z=xp_y-yp_x\end{cases}$$

    以l_x为例，有

    $$l_x^\dagger=(yp_z-zp_y)^\dagger=(p_z^\dagger y^\dagger-p_y^\dagger z^\dagger)=p_zy-p_yz=yp_z-zp_y=l_x$$

    同理可证l_y,l_z均是厄米的
  - $\boldsymbol{r}\cdot\boldsymbol{p}$

    $$\begin{aligned}(\boldsymbol{r}\cdot\boldsymbol{p})^\dagger&=(xp_x+yp_y+zp_z)^\dagger\\&=p_xx+p_yy+p_zz\\&=xp_x+yp_y+zp_z-3i\hbar\\&=\boldsymbol{r}\cdot\boldsymbol{p}-3i\hbar\end{aligned}$$

    因此该算符不是厄米的，但是可以添加常数$-1.5i\hbar$使之成为厄米的
  - $\boldsymbol{A}=\boldsymbol{p}\times\boldsymbol{l}$

$$\begin{aligned} A_x^\dagger &= (p_y x p_y - p_y y p_x - p_z z p_x + p_z x p_z)^\dagger \\ &= (p_y x p_y - p_x y p_y - p_x z p_z + p_z x p_z) \end{aligned}$$

由对易关系

$$\begin{aligned} [p_y y, p_x] &= p_y y p_x - p_x p_y y \\ &= \hbar^2 (\frac{\partial}{\partial x} + y\frac{\partial^2}{\partial x \partial y} - \frac{\partial}{\partial x} - y\frac{\partial^2}{\partial y \partial x}) \\ &= 0 \end{aligned}$$

同理可证$[p_z z, p_x] = 0$，因此原式化为

$$A_x^\dagger = (p_y x p_y - p_y y p_x - p_z z p_x + p_z x p_z) - 2i\hbar p_x = A_x - 2i\hbar p_x$$

因此该算符不是厄米的，但是可以添加算符$-i\hbar \boldsymbol{p}$使之成为厄米的

- $\boldsymbol{B} = \boldsymbol{r} \times \boldsymbol{l}$

$$\begin{aligned} B_x^\dagger &= (y l_z - z l_y)^\dagger \\ &= l_z y - l_y z \\ &= y l_z - z l_y - 2i\hbar x \\ &= B_x - 2i\hbar x \end{aligned}$$

因此该算符不是厄米的，但是可以添加算符$-i\hbar \boldsymbol{r}$使之成为厄米的

- **3.5**

$$\begin{aligned} A \cdot (B \times C) &= \sum_\gamma A_\gamma (B \times C)_\gamma \\ &= \sum_\gamma A_\gamma (\sum_{\alpha\beta} \epsilon_{\alpha\beta\gamma} B_\alpha C \gamma) \\ &= \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} A_\alpha B_\beta C_\gamma \end{aligned}$$

$$\begin{aligned} (A \times B) \cdot C &= \sum_\gamma (A \times B)_\gamma C_\gamma \\ &= \sum_\gamma (\sum_{\alpha\beta} \epsilon_\gamma A_\alpha B_\beta) C_\gamma \\ &= \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} A_\alpha B_\beta C_\gamma \end{aligned}$$

因此上述二者相等

$$
\begin{aligned}
[A\times(B\times C)]_\alpha &= \sum_{\beta\gamma}\epsilon_{\alpha\beta\gamma}A_\beta(B\times C)_\gamma \\
&= \sum_{\beta\gamma}\epsilon_{\alpha\beta\gamma}A_\beta(\sum_{ij}\epsilon_{ij\gamma}B_iC_j) \\
&= \sum_{\beta\gamma}\sum_{ij}\epsilon_{\alpha\beta\gamma}\epsilon_{ij\gamma}A_\beta B_iC_j \\
&= \sum_{ij\beta}A_\beta B_iC_j\sum_{\gamma}\epsilon_{\alpha\beta\gamma}\epsilon_{ij\gamma} \\
&= \sum_{ij\beta}A_\beta B_iC_j(\delta_{\alpha i}\delta_{\beta j}-\delta_{\alpha j}\delta_{\beta i}) \\
&= \sum_{ij\beta}A_\beta B_iC_\gamma\delta_{\alpha i}\delta_{\beta j}-\sum_{ij\beta}A_\beta B_iC_j\delta_{\alpha j}\delta_{\beta i} \\
&= \sum_{i\beta}A_\beta B_iC_j\delta_{\alpha i}-\sum_{j\beta}A_\beta B_\beta C_j\delta_{\alpha j} \\
&= \sum_{\beta}A_\beta(B_\alpha C_\beta-B_\beta C_\alpha) \\
&= A\cdot(B_\alpha C)-(A\cdot B)C_\alpha
\end{aligned}
$$

$[(A\times B)\times C]_\alpha = A\cdot(B_\alpha C)-A_\alpha(B\cdot C)$

- **3.6**

$$
\begin{aligned}
[F,A\cdot B] &= F\sum_i A_iB_i-\sum_i A_iB_iF \\
&= F\sum_i A_iB_i-\sum_i A_iFB_i+\sum_i A_iFB_i-\sum_i A_iB_iF \\
&= [F,A]\cdot B+A\cdot[F,B]
\end{aligned}
$$

$$
\begin{aligned}
[F,A\times B]_x &= [F,A_yB_z-A_zB_y] \\
&= FA_yB_z-FA_zB_y-A_yB_zF+A_zB_yF \\
&= (FA_yB_z-A_yFB_z-FA_zB_y+A_zFB_y)+(A_yFB_z-A_yB_zF-A_zFB_y+A_zB_yF) \\
&= [F,A]\times B+A\times[F,B]
\end{aligned}
$$

- **3.10**

(a)由于$\frac{1}{r}$是厄米的，因此有

$$
\begin{aligned}
p_r^\dagger &= \frac{1}{2}(\frac{1}{r}\boldsymbol{r}\cdot\boldsymbol{p}+\boldsymbol{p}\cdot\boldsymbol{r}\frac{1}{r})^\dagger \\
&= \frac{1}{2}((p_xx+p_yy+p_zz)\frac{1}{r}+\frac{1}{r}(xp_x+yp_y+zp_z)) \\
&= \frac{1}{2}(\boldsymbol{p}\cdot\boldsymbol{r}\frac{1}{r}+\frac{1}{r}\boldsymbol{r}\cdot\boldsymbol{p}) \\
&= p_r
\end{aligned}
$$

(b)球坐标系中散度表示为

$$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

因此可做如下化简

$$\begin{aligned}
p_r\psi &= -\frac{i\hbar}{2}(\frac{1}{r}\boldsymbol{r}\cdot\boldsymbol{\nabla} + \boldsymbol{\nabla}\cdot\boldsymbol{r}\frac{1}{r})\psi \\
&= -\frac{i\hbar}{2}(\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\psi)) \\
&= -i\hbar(\frac{\partial\psi}{\partial r} + \frac{\psi}{r}) \\
\Rightarrow p_r &= -i\hbar(\frac{\partial}{\partial r} + \frac{1}{r})
\end{aligned}$$

(c)

$$\begin{aligned}
[r, p_r]\psi &= (rp_r\psi - p_r r\psi) \\
&= i\hbar(-r\frac{\partial\psi}{\partial r} - \psi + \frac{\partial}{\partial r}(r\psi) + \psi) \\
&= i\hbar\psi \\
\Rightarrow [r, p_r] &= i\hbar
\end{aligned}$$

(d)

$$\begin{aligned}
p_r^2\psi &= -\hbar^2(\frac{\partial}{\partial r} + \frac{1}{r})(\frac{\partial\psi}{\partial r} + \frac{\psi}{r}) \\
&= -\hbar^2(\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r}) \\
\Rightarrow p_r^2 &= -\hbar^2(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}) = -\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}
\end{aligned}$$

(e)已知

$$\nabla^2 = \nabla\cdot(\nabla) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$p^2 = -\hbar^2\nabla^2$$

$$l^2 = -\hbar^2[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}]$$

带入d的结果即可证明

$$p^2 = \frac{1}{r^2} l^2 + p_r^2$$

- **3.12**

$$\begin{aligned}[\hat{x}, \hat{H}] &= [\hat{x}, \frac{p^2}{2m}] \\ &= \frac{1}{2m}(p[x,p] + [x,p]p) \\ &= \frac{i\hbar}{m} p\end{aligned}$$

因此动量的平均值转换成位置与哈密顿量的对易表示

$$\begin{aligned}\overline{p} &= \frac{m}{i\hbar}\overline{[\hat{x}, \hat{H}]} \\ &= \frac{m}{i\hbar}\int \psi^*[\hat{x}, \hat{H}]\psi d\epsilon \\ &= \frac{m}{i\hbar}\int E(\psi^* \hat{x}\psi - \psi^* \hat{x}\psi) d\epsilon \\ &= 0\end{aligned}$$

- **3.14** l_z的本征态可以表示为

$$\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

$$L_z = m\hbar, m = 0, \pm 1, \pm 2, \cdots$$

故有

$$\begin{aligned}\overline{l_z} &= \int \psi_m^* l_z \psi_m d\tau \\ &= \frac{m\hbar}{2\pi}\int_0^{2\pi} d\phi \\ &= m\hbar\end{aligned}$$

由对易关系$[l_z, l_x] = i\hbar l_y$，有

$$\begin{aligned}\overline{l_y} &= \frac{1}{i\hbar}\overline{[l_z, l_x]} \\ &= \frac{1}{i\hbar}\int \psi_m^*[l_z, l_x]\psi_m d\tau \\ &= \frac{m}{i}\int (\psi_m^* l_x \psi_m - \psi_m^* l_x \psi_m) d\tau \\ &= 0\end{aligned}$$

同理，$\overline{l_x} = 0$

- **3.16**

(a)Y_{lm}是$\hat{l_z}$的本征态，本征值为$m\hbar$，因此在本题的波函数情形下，l_z可能测得$\hbar$和0，概率分别为$|c_1^2|$和$|c_2^2|$

(b)Y_{lm}也是l^2的本征态，本征值为$l(l+1)\hbar^2$，本题中可能测得$2\hbar^2$和$6\hbar^2$，概率分别为$|c_1|^2$和$|c_2|^2$

- **12题**

$$
\begin{aligned}
H\psi &= E\psi \\
[\frac{1}{2I_1}(l_x^2+l_y^2+\frac{1}{2}l_z^2)+\frac{1}{2I_2}l_z^2]\psi &= E\psi \\
\frac{1}{2I_1}l^2\psi+\frac{2I_1-I_2}{4I_1I_2}l_z^2\psi &= E\psi
\end{aligned}
$$

在球坐标系中，两算符的本征函数表示分别如下

$$
\begin{cases}
l^2Y_{lm}=l(l+1)\hbar^2Y_{lm} \\
l_z^2Y_{lm}=m^2\hbar^2Y_{lm}
\end{cases}
$$

因此能量本征值

$$
E=\frac{l(l+1)\hbar^2}{2I_1}+\frac{2I_1-I_2}{4I_1I_2}m^2\hbar^2
$$

相应的本征函数

$$
Y_{lm}=(-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}
$$